

Gaussian Vortex Approximation to the Instanton Equations of two-dimensional Turbulence

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We investigate two-dimensional turbulence within the Instanton formalism which determines the most probable field in a stochastic classical field theory starting from the Martin-Siggia-Rose path integral. We perform an approximate analysis of these equations based on a variational ansatz using elliptical vortices. The result are evolution equations for the positions and the shapes of the vortices. We solve these ordinary differential equations numerically. The extremal action for the two-point statistics is determined by the merging of two elliptical vortices. We discuss the relationship of this dynamical system to the inverse cascade process of two-dimensional turbulence.

Keywords: Instanton, Turbulence, Elliptic Vortices, Two-dimensional, Martin-Siggia-Rose, Inverse Cascade

I. INTRODUCTION

Over one and a half centuries after the discovery of the Navier-Stokes equations, the theoretical understanding of turbulent flows remains quite limited although phenomenological theories (see [1]) capture the main characteristics of turbulent flows quite well. However, no generally accepted derivation of the phenomenological theories from the basic hydrodynamical equations has been possible. The challenge consists in connecting dynamical systems theory with nonequilibrium statistical physics [2].

The study of two-dimensional turbulence is motivated by the fact that whenever one spatial direction is significantly constrained (e.g. layers in the atmosphere or all kinds of surface dynamics) the flow can be described as quasi two-dimensional. In a two-dimensional space, the behavior of turbulent flows differs significantly from the three-dimensional case. These differences arise due to an additional conservation law for the enstrophy which leads to a new phenomenon called the inverse cascade discovered in 1967 by Kraichnan [3]. Numerically, a stationary inverse energy cascade can be generated by a stochastic forcing of the flow at small scales while dissipating energy at large scales.

The stochastically forced vorticity equation of two-dimensional turbulence can be treated as a stochastic classical field theory using the Martin-Siggia-Rose path integral formalism [4]. Path integrals may be treated either perturbatively using renormalized perturbation theory or within the Instanton formalism by an expansion of the field around the most probable field configuration. This formalism has been applied by several groups to the Burgers equation (see [5–8]), to the problem of turbulent advection and to estimate the tails of the velocity probability distribution of fully developed turbulence [9]. We also refer the readers to earlier works by Falkovich [10] and Moriconi [11].

The purpose of the present contribution is to investigate the dynamics of the Instanton equations for the inverse cascade in two-dimensional turbulent flows. Thereby, we do not focus on the concrete evaluation of the extremal action and the shape of the probability distribution. As has been pointed out by [9], a reasonable estimation of the probability distribution has to take into account fluctuations around the Instanton. We rather address the question whether one can identify signatures of the mechanism underlying the inverse cascade already on the basis of the N -point Instantons.

Our approach consists of an ansatz for the Instanton field based on a superposition of elliptical, deformable vortices. We derive the most probable evolution of these vortices by extremalizing the Martin-Siggia-Rose [4] action (see eq. (5)). This ansatz is motivated by the observation that the Instanton equations in the inviscid limit and in the presence of small scale forcing are related to the dynamics of point vortices (see [12–14]).

We design a numerical procedure to solve the resulting evolution equations for the positions and the elliptical shapes explicitly for two vortices. This gives the two-point statistics in the Instanton approximation. Our aim here is not the complete determination of this action, which is a project for future research. The aim of the present paper is to investigate the deterministic dynamics of the Instanton equations. Our variational ansatz using elliptical point vortices demonstrates that the action is essentially determined by the interaction of two elliptical vortices. Thereby, the elongation of the main axis of the elliptical vortices plays a crucial role. This elongation leads in the case of the interaction of vortices with like-signed circulation to an accelerated relative motion (see also [15]).

The structure of this paper is as follows. We start with the investigation of the dynamics of Instanton point vortices which do not exhibit an inverse cascade. We present a generalized model that takes into account the deformation of vortices due to strain and calculate the interaction of two vortices numerically. We discuss the relationship of this model to the inverse cascade process.

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II. INSTANTON EQUATIONS

We start from the two-dimensional vorticity equation for incompressible fluids with $\omega = \nabla \times \mathbf{u}$

$$\partial_t \omega(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t) \cdot \nabla \omega(\mathbf{x}, t) - \nu \Delta \omega(\mathbf{x}, t) = f(\mathbf{x}, t) \quad (1)$$

where $f(\mathbf{x}, t)$ denotes an external Gaussian white noise forcing which is specified by

$$\langle f(\mathbf{x}, t) f(\mathbf{x}', t') \rangle = Q(\mathbf{x} - \mathbf{x}') \delta(t - t'). \quad (2)$$

The velocity field can be expressed in terms of the vorticity field via the Biot-Savart law according to

$$\mathbf{u}(\mathbf{x}, t) = \int \frac{d^2 x'}{2\pi} \mathbf{e}_z \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^2} \omega(\mathbf{x}', t). \quad (3)$$

Within the Martin-Siggia-Rose formalism [4], all moments are derived from the partition functional which is given by the path integral

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\omega \mathcal{D}\hat{\omega} e^{iS + \int d^2 x dt [\eta(\mathbf{x}, t) \omega(\mathbf{x}, t)]} \\ &= \int \mathcal{D}\omega \mathcal{D}\hat{\omega} e^{i\tilde{S}} \end{aligned} \quad (4)$$

with the extended Martin-Siggia-Rose action

$$\begin{aligned} \tilde{S} &= \int d^2 x dt \hat{\omega}(\mathbf{x}, t) \left[\partial_t \omega(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t) \cdot \nabla \omega(\mathbf{x}, t) \right. \\ &\quad \left. - \nu \Delta \omega(\mathbf{x}, t) + \frac{i}{2} \int d^2 x' Q(\mathbf{x} - \mathbf{x}') \hat{\omega}(\mathbf{x}', t) \right] \\ &\quad - i \int d^2 x dt \eta(\mathbf{x}, t) \omega(\mathbf{x}, t). \end{aligned} \quad (5)$$

The partition functional \mathcal{Z} can be related to the Laplace transform of the N -point vorticity probability distribution by evaluating \mathcal{Z} for the field $\eta(\mathbf{x}, t) = \sum_{i=1}^n \alpha_i \delta(\mathbf{x} - \mathbf{x}_i)$.

The Instanton equations are derived by performing a variation of the action (5) with respect to the vorticity ($\delta S / \delta \omega = 0$) and the auxiliary field ($\delta S / \delta \hat{\omega} = 0$). This procedure yields two coupled partial differential equations that describe the evolution of the most probable field configuration

$$\begin{aligned} \partial_t \omega &= -\mathbf{u} \cdot \nabla \omega + \nu \Delta \omega - i \int d^2 x' Q(\mathbf{x} - \mathbf{x}') \hat{\omega}(\mathbf{x}', t) \\ \partial_t \hat{\omega} &= - \int \frac{d^2 x'}{2\pi} \hat{\omega}(\mathbf{x}', t) \frac{\mathbf{e}_z \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2} \cdot \nabla \omega(\mathbf{x}', t) \\ &\quad - \mathbf{u} \cdot \nabla \hat{\omega} - \nu \Delta \hat{\omega} + \eta(\mathbf{x}, t). \end{aligned} \quad (6)$$

The extremal action is then given by

$$iS_{\text{extr}} = -\frac{1}{2} \int d^2 x' \hat{\omega}(\mathbf{x}, t) Q(\mathbf{x} - \mathbf{x}') \hat{\omega}(\mathbf{x}', t). \quad (7)$$

The equation for ω has to be evaluated forward in time from $t^* < 0$ to 0, whereas the equation for $\hat{\omega}$ has to be

solved backward in time from 0 to $t^* < 0$. This is in accordance with the different sign of viscosity. For the auxiliary field we have the initial condition

$$\hat{\omega}(\mathbf{x}, 0) = \sum_{i=1}^n \alpha_i \delta(\mathbf{x} - \mathbf{x}_i) \quad (8)$$

which results from considering the n -point statistics of the field at spatial points \mathbf{x}_i at time $t = 0$ by setting $\eta = \delta(t) \sum_{i=1}^n \alpha_i \delta(\mathbf{x} - \mathbf{x}_i)$.

III. ZERO-VISCOSITY LIMIT

In order to motivate our further treatment, let us consider the evolution equation for the auxiliary field in the limit of vanishing viscosity under the neglect of the third term on the left-hand side of equation (1)

$$\partial_t \hat{\omega} + \mathbf{u} \cdot \nabla \hat{\omega} = \sum_{i=1}^n \alpha_i \delta(\mathbf{x} - \mathbf{x}_i) \delta(t). \quad (9)$$

The solution to this equation is given by a sum over delta-functions

$$\hat{\omega}(\mathbf{x}, t) = \sum_{i=1}^n \alpha_i \delta(\mathbf{x} - \mathbf{X}(\mathbf{x}_i, t)) \quad (10)$$

where $\mathbf{X}(\mathbf{x}_i, t)$ denotes the Lagrangian path (for $t \leq 0$) of a passive particle moving in the velocity field $\mathbf{u}(\mathbf{X}, t)$ ending at \mathbf{x}_i at time $t = 0$ governed by the evolution equation

$$\frac{\partial}{\partial t} \mathbf{X}(\mathbf{x}, t) = \mathbf{u}(\mathbf{X}(\mathbf{x}, t), t) \quad (11)$$

with the velocity field related to the field $\omega(\mathbf{x}, t)$ by Biot-Savart's law.

The temporal evolution for the Instanton field $\omega(\mathbf{x}, t)$ in the zero-viscosity limit is then given by

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = - \sum_{i=1}^n Q(\mathbf{x} - \mathbf{X}(\mathbf{x}_i, t)) \alpha_i. \quad (12)$$

This equation can also be solved using the Lagrangian path $\mathbf{X}(\mathbf{x}, t)$ by evaluating

$$\begin{aligned} \omega(\mathbf{X}(\mathbf{x}, t)) &= \\ &= - \sum_{i=1}^n \alpha_i \int_{t^*}^t dt' Q(\mathbf{X}(\mathbf{x}, t') - \mathbf{X}(\mathbf{x}_i, t')). \end{aligned} \quad (13)$$

For small-scale stirring we can approximate the correlation $Q(\mathbf{r})$ by a δ -function.

Finally, we have to determine the velocity field, which is obtained by a combination of Biot-Savart's law and the representation of the vorticity field

$$\mathbf{u}(\mathbf{x}, t) = \sum_{i=1}^n \alpha_i \left[\int_{t^*}^t dt' q \mathbf{e}_z \times \frac{\mathbf{x} - \mathbf{X}(\mathbf{x}_i, t')}{2\pi |\mathbf{x} - \mathbf{X}(\mathbf{x}_i, t')|^2} \right]. \quad (14)$$

For the sake of simplicity, we approximated $Q(\mathbf{x} - \mathbf{x}')$ by a delta function, $q\delta(\mathbf{x} - \mathbf{x}')$. We obtain an extension of the usual point vortex dynamics by the fact that the circulations is time dependent $\propto (t - t^*)\alpha_i q$.

This dependence of the initial time t^* can be taken into account by a time shift $\tau = t - t^*$. The Instanton point vortex dynamics is given by $(\mathbf{x}_i(\tau) = \mathbf{X}(\mathbf{x}_i, \tau + t^*))$

$$\frac{d}{d\tau}\mathbf{x}_i = \tau \sum_{j=1}^n \mathbf{e}_z \times \frac{\mathbf{x} - \mathbf{x}_j(\tau)}{2\pi|\mathbf{x} - \mathbf{x}_j(\tau)|^2}. \quad (15)$$

The transformation $\tau' = \frac{1}{2}\tau^2$ yields the usual point vortex dynamics. Therefore, we conclude that the inviscid Instanton point vortex dynamics is similar to the usual point vortex dynamics in an accelerated timeframe which consists of replacing the τ -dependence by a τ^2 dependence.

On the basis of the point vortex approximation, there is no mechanism which could be related to the inverse cascade.

IV. DEFORMABLE VORTICES

In this section we sketch a generalization taking into account the finite extension of the vortices as well as a deformation of the vortices due to shear generated by distant vortices. This will motivate our approximate treatment of the Instanton equations.

We start with the auxiliary field, which for $t \rightarrow 0$ actually has to tend to a superposition of δ -functions, i.e. to a superposition of point vortices. For negative times, viscosity will broaden the vortices whereas presence of the advective term and the type of vortex stretching term will lead to a deformation of the circular shape. Thus, the straightforward extension is an ansatz in terms of elliptical point vortices

$$\hat{\omega}(\mathbf{x}, t) = \sum_{j=1}^m \frac{\hat{\Gamma}_j(t)}{2\pi\sqrt{\det \hat{C}_j}} e^{-\frac{1}{2}(\mathbf{x} - \hat{\mathbf{x}}_j(t))\hat{C}_j^{(-1)}(\mathbf{x} - \hat{\mathbf{x}}_j(t))}. \quad (16)$$

From the condition (8) we conclude that

$$\hat{\Gamma}_j(0) = \alpha_j, \hat{C}_j(0) = 0, \text{ and } \hat{\mathbf{x}}_j(0) = \mathbf{x}_j. \quad (17)$$

A similar ansatz is performed for the Instanton field $\omega(\mathbf{x}, t)$

$$\omega(\mathbf{x}, t) = \sum_{j=1}^n \frac{\Gamma_j(t)}{2\pi\sqrt{\det C_j}} e^{-\frac{1}{2}(\mathbf{x} - \mathbf{x}_j(t))C_j(t)^{(-1)}(\mathbf{x} - \mathbf{x}_j(t))}. \quad (18)$$

The initial condition for $t^* \rightarrow -\infty$, $\omega(\mathbf{x}, t) = 0$ is defined by $\Gamma_j(t^*) = 0$, $C_j(t^*)$ and the initial locations $\mathbf{x}_j(t^*)$. The initial condition $\omega(\mathbf{x}, t^*) = 0$ can be realized by $\Gamma_j(t^*) = 0$.

It is evident that we could extend our ansatz by including, besides the elliptical ansatz, higher-order terms. Such an expansion with an additional averaging over the higher-order contributions is needed in order to go beyond the bare Instanton approximation. This program would lead to an effective action in much the same way as has been demonstrated by Falkovich et al. [9] in their treatment of the statistics of the vorticity increment of the direct cascade. They separated the fields into an axisymmetric and a nonaxisymmetric part, where the nonaxisymmetric fluctuations were integrated out to obtain an effective action for the axisymmetric part, which, in turn was treated in the Instanton approximation. We conjecture that such an effective action could be characterized by a modified noise correlation $Q(\mathbf{x} - \mathbf{x}')$ as well as a modified Biot-Savart law, $\mathbf{u}(\mathbf{x}, t) = \mathbf{e}_z \times \nabla F(-\Delta)\omega(\mathbf{x}, t)$, where $F(-\Delta)$ is a function of the Laplacian.

In the present article we will be content with the elliptic vortex approximation. Our main point here is that the behavior of the elliptical vortices depends on the signs of Γ_j and, hence, on the sign of α_j . We will explicitly demonstrate that two like-signed vortices will attract each other due to this deformation.

V. ACTION FOR ELLIPTIC VORTICES

In this section we consider the Martin-Siggia-Rose action for a field which consists of a superposition of N elliptical vortices. Each vortex is determined by its location $\mathbf{x}_i(t)$, its amplitude $\Gamma_i(t)$ and a matrix $C_i(t)$ determining the orientation and the semi-axis of the elliptical vortices. Similar quantities are introduced for the auxiliary field. By introducing these collective variables, the Instanton equations, which are partial differential equations, are approximated by ordinary differential equations, which are the Euler-Lagrange equations for the MSR-action, written in collective variables. These equations turn out to be extensions of the inviscid point vortex Instantons. The inclusion of the elliptical shape of the vortices, however, will lead to a modified vortex dynamics.

As a starting point we consider the Martin-Siggia-Rose action (eq. (5)) in Fourier space

$$S = \int d^2k dt \hat{\omega}(-\mathbf{k}, t) \left\{ \partial_t \omega(\mathbf{k}, t) + \nu k^2 \omega(\mathbf{k}, t) - i\mathbf{k} \cdot \int d^2k' \mathbf{u}(\mathbf{k}') \omega(\mathbf{k} - \mathbf{k}', t) \omega(\mathbf{k}', t) \right\} + \frac{i}{2} \int d^2k dt \hat{\omega}(-\mathbf{k}, t) Q(\mathbf{k}) \hat{\omega}(\mathbf{k}, t) \quad (19)$$

where we defined $\mathbf{u}(\mathbf{k}) = i \frac{\mathbf{e}_z \times \mathbf{k}}{4\pi^2 k^2}$. We make an elliptic ansatz for the vorticity and the auxiliary field according

to

$$\begin{aligned}\omega(\mathbf{k}, t) &= \sum_{i=1}^n \Gamma_i e^{i\mathbf{k} \cdot \mathbf{x}_i - \frac{1}{2} \mathbf{k} C_i \mathbf{k}} \equiv \sum_{i=1}^n \Gamma_i \omega_i(\mathbf{k}, t) \\ \hat{\omega}(\mathbf{k}, t) &= i \sum_{i=1}^m \hat{\Gamma}_i e^{i\mathbf{k} \cdot \hat{\mathbf{x}}_i - \frac{1}{2} \mathbf{k} \hat{C}_i \mathbf{k}} \equiv i \sum_{i=1}^m \hat{\Gamma}_i \hat{\omega}_i(\mathbf{k}, t).\end{aligned}\quad (20)$$

C_i and \hat{C}_i are 2×2 symmetric matrices and Γ_i and $\hat{\Gamma}_i$ are the circulations of each vortex or elliptic structure in the auxiliary field.

Here, we assume that the quantities $\Gamma_i(t)$, $\mathbf{x}_i(t)$, $C_i(t)$ as well as the corresponding quantities for the auxiliary field depend on time. Extremalization of the MSR-action will lead us to an approximation of the Instanton equations in terms of a coupled set of ordinary differential equations. As we have discussed in the previous section, we include the deformation of the vortices since this deformation seems to play a major role for the energy transport in the inverse cascade.

We have to specify the boundary conditions for the fields with respect to time. The conditions for the auxiliary field are easily formulated. From the representation (8) we obtain

$$\hat{\mathbf{x}}_i(0) = \mathbf{x}_i \text{ and } \hat{C}_i(0) = 0. \quad (21)$$

The initial condition for the field $\omega(\mathbf{x}, t)$ has to be formulated for $t \rightarrow -\infty$, in particular we have $\omega(\mathbf{x}, t \rightarrow -\infty) = 0$. As we will see below, this can be achieved by putting $C_i(-\infty) = 0$, i.e. a infinitely extended vortex. Another possibility would be to set $\Gamma_i(t^*) = 0$ for a large negative time t^* . The reason will become more evident below, when we formulate the Euler-Lagrange equation corresponding to the MSR-action for the elliptical vortices.

We plug the ansatz into the action (19) and integrate the action with respect to \mathbf{k} which yields

$$\begin{aligned}S &= \sum_{i=1}^m \sum_{j=1}^n \int dt \hat{\Gamma}_i \Gamma_j \left\{ \frac{\dot{\Gamma}_j}{\Gamma_j} - \dot{\mathbf{x}}_j \cdot \hat{\nabla}_i + \frac{1}{2} \hat{\nabla}_i \dot{C}_j \hat{\nabla}_i \right. \\ &\quad \left. - i \sum_{k=1}^n \Gamma_k A_{jk}(\hat{\nabla}_i) - \nu \hat{\nabla}_i \mathcal{I} \hat{\nabla}_i + \frac{i}{2} \frac{\dot{\Gamma}_j}{\Gamma_j} K_j(\hat{\nabla}_i) \right\} W_{ij},\end{aligned}\quad (22)$$

where $\hat{\nabla}_i$ denotes $\nabla_{\hat{\mathbf{x}}_i}$ and we defined

$$\begin{aligned}A_{jk}(\hat{\nabla}_i) &= i \int d^2 k' e^{(\mathbf{x}_j - \mathbf{x}_k) \cdot \hat{\nabla}_i + \frac{1}{2} \hat{\nabla}_i (C_j + C_k) \hat{\nabla}_i} \\ &\quad \times e^{\frac{i}{2} \hat{\nabla}_i C_j \mathbf{k}' + \frac{i}{2} \mathbf{k}' C_j \hat{\nabla}_i} \mathbf{u}(i \hat{\nabla}_i) \cdot \hat{\nabla}_i, \\ K_j(\hat{\nabla}_i) &= i Q(i \hat{\nabla}_i) e^{-(\hat{\mathbf{x}}_j - \mathbf{x}_j) \cdot \hat{\nabla}_i + \frac{1}{2} \hat{\nabla}_i (\hat{C}_j - C_j) \hat{\nabla}_i}, \\ W_{ij} &= i \frac{2\pi}{\sqrt{\det[\hat{C}_i + C_j]}} \\ &\quad \times e^{-\frac{1}{2} (\hat{\mathbf{x}}_i - \mathbf{x}_j) (\hat{C}_i + C_j)^{-1} (\hat{\mathbf{x}}_i - \mathbf{x}_j)}.\end{aligned}\quad (23)$$

Equation (22) is the action for elliptic vortices and elliptic structures in the auxiliary field according to the ansatz from eq. (20). We will derive the evolution equations for the parameters in the ansatz by applying a variation according to the Euler-Lagrange equations. To this end, we make the following assumptions.

1. We assume that we have the same number $n = m = N$ of elliptic vortices and elliptic structures in the auxiliary field.
2. The elliptic structures in the auxiliary field are positioned such that the i -th vortex is always close to the i -th elliptic structure, which means

$$|\mathbf{x}_i(t) - \hat{\mathbf{x}}_i(t)| \ll \text{Size of vortices}. \quad (24)$$

3. The vortices and elliptic structures are isolated, hence their overlap can be omitted. This yields $\forall i \neq j$

$$|\mathbf{x}_i(t) - \mathbf{x}_j(t)| \gg \text{Size of vortices} \quad (25a)$$

and $\forall i \neq j$

$$|\hat{\mathbf{x}}_i(t) - \hat{\mathbf{x}}_j(t)| \gg \text{Size of elliptic structures}. \quad (25b)$$

Hence, we only account for the diagonal terms in eq. (22). Based on this approximation we pursue the investigation of the action

$$\begin{aligned}S &= \sum_{i=1}^N \int dt \hat{\Gamma}_i \Gamma_i \left\{ \frac{\dot{\Gamma}_i}{\Gamma_i} - \dot{\mathbf{x}}_i \cdot \hat{\nabla} + \frac{1}{2} \hat{\nabla} \dot{C}_i \hat{\nabla} \right. \\ &\quad \left. - i \sum_{j=1}^N \Gamma_j A_{ij}(\hat{\nabla}) - \nu \hat{\nabla} \mathcal{I} \hat{\nabla} + \frac{i}{2} \frac{\dot{\Gamma}_i}{\Gamma_i} K_i(\hat{\nabla}) \right\} W_{ii},\end{aligned}\quad (26)$$

where we have set $i = j$ and then renamed k to j .

VI. EVOLUTION EQUATIONS FOR COLLECTIVE COORDINATES

We formulate the evolution equations for our Instanton model of elliptical vortices and the corresponding auxil-

iary field

$$\begin{aligned}
\dot{\Gamma}_i &= q\hat{\Gamma}_i, \\
\dot{\mathbf{x}}_i &= \sum_{j=1}^N \Gamma_j \mathbf{U}_{ij}(\mathbf{x}_i - \mathbf{x}_j) + q \frac{\hat{\Gamma}_i}{\Gamma_i} (\hat{\mathbf{x}}_i - \mathbf{x}_i(t)), \\
\dot{C}_i &= \sum_{j=1}^N \Gamma_j [S_{ij}(\mathbf{x}_i - \mathbf{x}_j) C_i + C_i S_{ij}^T(\mathbf{x}_i - \mathbf{x}_j)] \\
&\quad + q \frac{\hat{\Gamma}_i}{\Gamma_i} (\tilde{Q} + \hat{C}_i - C_i) + 2\nu \mathcal{I}, \\
\dot{\hat{\Gamma}}_i &= 0, \\
\dot{\hat{\mathbf{x}}}_i &= \sum_{j=1}^N \Gamma_j \hat{\mathbf{U}}_{ij}(\hat{\mathbf{x}}_i - \mathbf{x}_j), \\
\dot{\hat{C}}_i &= \sum_{j=1}^N \Gamma_j [\hat{S}_{ij}(\hat{\mathbf{x}}_i - \mathbf{x}_j) \hat{C}_i + \hat{C}_i \hat{S}_{ij}^T(\hat{\mathbf{x}}_i - \mathbf{x}_j)] \\
&\quad - 2\nu \mathcal{I}
\end{aligned} \tag{27}$$

in collective coordinates as the main result of our approach. Note that we accounted for the previously mentioned assumptions in the derivation of these equations which can be found in the appendix A.

These equations allow for a subset of solutions for which the locations of the vortices of both fields $\mathbf{x}_i(t)$ and $\hat{\mathbf{x}}_i(t)$ coincide. We will discuss this case later.

VII. TWO-POINT INSTANTON

We solve the evolution equations explicitly for two vortices with same-signed circulations. In accordance with the assumptions made earlier, we allow for two elliptically shaped structures in the auxiliary field which have to tend to infinitely small objects once time t tends to 0. This means that at time 0 the auxiliary field consists of a superposition of two delta functions (see eq. (8)) and hence is connected to the two-point statistics of the vorticity field. We call the corresponding Instanton the two-point Instanton which provides an insight to the interaction of two vortices.

We want to derive the dynamics of the two-point Instanton to understand the underlying vortex dynamics. To this end, we expand the advection and deformation and make use of an iterative calculation scheme to obtain the solution for the vorticity and auxiliary field.

Let us consider the calculations in some more detail. We want to calculate the evolution of two elliptic vortices and two elliptic structures governed by the evolution equations (27).

To this end, we expand the advection for $i \neq j$ [19]

$$\mathbf{U}_{ij}(\mathbf{r}) \approx \left[1 + \frac{1}{2} \nabla_r^T (C_i + C_j) \nabla_r \right] \mathbf{e}_z \times \frac{\mathbf{r}}{2\pi|\mathbf{r}|^2} \tag{28}$$

until first order in $C_i + C_j$ and define

$$\mathbf{V}(\mathbf{x}) = \frac{1}{2\pi r^2} \begin{pmatrix} -y \\ x \end{pmatrix}. \tag{29}$$

We can write $\mathbf{U}_{ij}(\mathbf{r})$ with $C_{\mu\nu}^{(ij)} = \frac{1}{2}(C_{\mu\nu}^{(i)} + C_{\mu\nu}^{(j)})$ as

$$\mathbf{U}_{ij}(\mathbf{r}) = \mathbf{V}(\mathbf{r}) + \sum_{\alpha,\beta=1}^2 C_{\alpha\beta}^{(ij)} \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial x_\beta} \mathbf{V}(\mathbf{r}) \tag{30}$$

where $C_{\mu\nu}^{(i)}$ denotes the component $\mu\nu$ of the matrix C_i .

For the deformation we have to evaluate

$$S_{ij}(\mathbf{r}) = \left[1 + \frac{1}{2} \nabla_r^T (C_i + C_j) \nabla_r \right] \mathbf{e}_z \times \frac{\mathbf{r}}{2\pi|\mathbf{r}|^2} \nabla_r^T \tag{31}$$

which we can write as

$$S_{ij}(\mathbf{r}) = M(\mathbf{r}) + \sum_{\alpha,\beta=1}^2 C_{\alpha\beta}^{(ij)} \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial x_\beta} M(\mathbf{r}) \tag{32}$$

with the matrix

$$M(\mathbf{x}) = \frac{1}{2\pi r^4} \begin{pmatrix} 2xy & 2y^2 - r^2 \\ -2x^2 + r^2 & -2xy \end{pmatrix}. \tag{33}$$

An analogue calculation for the auxiliary field yields

$$\hat{\mathbf{U}}_{ij}(\mathbf{r}) = \mathbf{V}(\mathbf{r}) + \sum_{\alpha,\beta=1}^2 \tilde{C}_{\alpha\beta}^{(ij)} \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial x_\beta} \mathbf{V}(\mathbf{r}) \tag{34}$$

and

$$\hat{S}_{ij}(\mathbf{r}) = M(\mathbf{r}) + \sum_{\alpha,\beta=1}^2 \tilde{C}_{\alpha\beta}^{(ij)} \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial x_\beta} M(\mathbf{r}) \tag{35}$$

where $\tilde{C}_{\mu\nu}^{(ij)}$ denotes $\hat{C}_{\mu\nu}^{(i)} + C_{\mu\nu}^{(j)}$ and $\mathbf{r} \rightarrow \hat{\mathbf{x}}_i - \mathbf{x}_j$. The auxiliary field satisfies the initial condition $\hat{\omega}(\mathbf{x}, 0) = \sum_{i=1}^2 \alpha_i \delta(\mathbf{x} - \hat{\mathbf{x}}_i)$ according to eq. (8) which corresponds to two point vortices. The vorticity field is fixed at a time $t^* < 0$. Because the initial conditions for the two fields have to be fixed at different times, a straight forward calculation is not feasible.

We solve the equations for two vortices and two elliptic structures in the auxiliary field using an iterative technique. We start from the well known solution for two point vortices (this emerges in the limit of vanishing C_i and \hat{C}_i) located at \mathbf{x}_1 and \mathbf{x}_2 or $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ respectively with $\mathbf{x}_i(t) = \hat{\mathbf{x}}_i(t)$ for $i = 1, 2$. The initial condition for the auxiliary field consists of two point vortices located at $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ so that one can argue that $\hat{\omega}$ will behave similar to the motion of a two point vortex system. Now we calculate the evolution of the vorticity field ω according to the set of equations (A9) in the limit $\hat{C}_i \rightarrow 0$. Hence, we have $\hat{\mathbf{x}}_i(t) = U(\vartheta t) \mathbf{x}_i(0)$ where U denotes the two-dimensional rotation matrix $SO(2)$ and $\vartheta = (\Gamma_1 + \Gamma_2)/(2\pi d^2)$. We use the approximations for

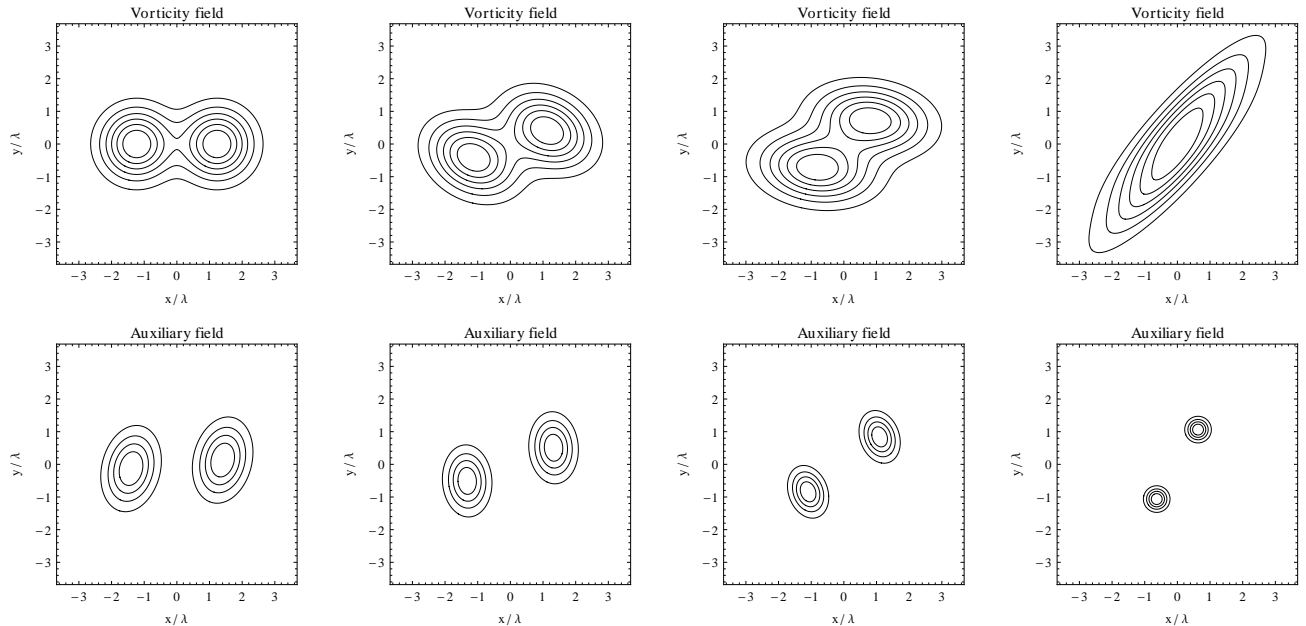


Figure 1: Evolution of two vortices of same-signed circulation and their corresponding auxiliary fields. The vortices main axes become elongated and the vortices approach each other. Time increases from left to right.

the advection \mathbf{U}_{ij} and deformation S_{ij} from equations (30) and (32) to calculate $\mathbf{x}_i(t)$ and $C_i(t)$. The ordinary differential equations can be solved with an explicit Runge-Kutta method.

Now we can calculate the auxiliary field $\hat{\omega}$ for $\hat{C}_i \neq 0$ if we plug in the previously calculated solution for the vorticity field given by $\Gamma_i(t)$, $\mathbf{x}(t)$ and $C_i(t)$. Because of $\partial_t \hat{\Gamma}_i \approx 0$ the circulation $\Gamma_i(t)$ grows linearly in time according to $q\hat{\Gamma}_i(t - t^*) + \Gamma_i(t^*)$. With the obtained solution for the elliptic structures in the auxiliary field we can calculate a more precise solution of the vorticity field and successively repeat these steps. Due to the smallness of the coupling parameter q compared to the circulations ($q \approx 0.05$ where we chose the circulations in the order of unity) this iterative procedure converges quickly.

The result for same-signed vortices is shown in figure 1. Initially circular shaped vortices are deformed with time, forming long elliptic structures. These structures then approach each other which is shown in figure 2, where we plotted the distance between the centers of the vortices.

VIII. REDUCED EQUATIONS

We want to discuss the relationship of the elliptic model described by the set of evolution equations (27) to the inverse cascade in more detail. To this end, we formulate a reduced version of the evolution equations. We perform the following approximations:

1. The position of the center of each vortex is equal to the center of each structure in the auxiliary field

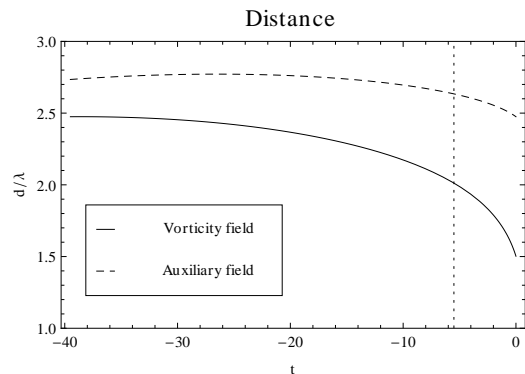


Figure 2: Evolution of the distance between the centers of the vortices (solid line) and the centers of the elliptic structures in the auxiliary field (dashed line). One sees the tendency of the vortices to approach each other rapidly once the elongation becomes dominant (vertical dotted line).

at each time t between t^* and 0. This means

$$\mathbf{x}_i = \hat{\mathbf{x}}_i \quad \forall t \in [t^*, 0] \quad (36)$$

and hence the term $q\frac{\hat{\Gamma}_i}{\Gamma_i}(\hat{\mathbf{x}}_i - \mathbf{x}_i(t))$ in (27) vanishes.

2. The structures in the auxiliary field are infinitely small and can be described by δ functions. In terms of the collective coordinates, this means

$$\hat{C}_i = 0 \quad \forall t \in [t^*, 0]. \quad (37)$$

As a consequence, the coupling term in the evolu-

tion equation for C_i reduces to

$$q \frac{\hat{\Gamma}_i}{\Gamma_i} (\tilde{Q} + \hat{C}_i - C_i) \rightarrow q \frac{\hat{\Gamma}_i}{\Gamma_i} (\tilde{Q} - C_i) \quad (38)$$

and hence the evolution of C_i becomes decoupled.

We obtain a reduced set of evolution equations for the vorticity field that implicitly contains the evolution of the auxiliary field given the above approximations. One can think of it as an enslavement process where the evolution of the point vortex like auxiliary field follows instantaneously the centers of the elliptic vortices. It shall also be noted that the approximation $\dot{\hat{C}}_i(t) = 0$ is consistent with the initial condition at time $t = 0$. The resulting equations for the vorticity field are

$$\begin{aligned} \dot{\hat{\Gamma}}_i &= q \hat{\Gamma}_i, \\ \dot{\mathbf{x}}_i &= \sum_{j=1}^N \Gamma_j \mathbf{U}_{ij}(\mathbf{x}_i - \mathbf{x}_j), \\ \dot{C}_i &= \sum_{j=1}^N \Gamma_j [S_{ij}(\mathbf{x}_i - \mathbf{x}_j) C_i + C_i S_{ij}^T(\mathbf{x}_i - \mathbf{x}_j)] \\ &\quad + q \frac{\hat{\Gamma}_i}{\Gamma_i} (\tilde{Q} - C_i) + 2\nu \mathcal{I}. \end{aligned} \quad (39)$$

Given the solution to (39) the evolution of the auxiliary field is well known. The circulation $\hat{\Gamma}_i$ is constant, the position $\hat{\mathbf{x}}_i$ is equal to \mathbf{x}_i and the matrix \hat{C}_i is zero. Nevertheless, the effect of the auxiliary field is still present and taken into account by the term $q \frac{\hat{\Gamma}_i}{\Gamma_i} (\tilde{Q} - C_i)$.

It is interesting to mention that (39) is equivalent to the equations for elliptic vortices presented by Friedrich and Friedrich in [16] from which they derived the rotor model. The term $q \frac{\hat{\Gamma}_i}{\Gamma_i} (\tilde{Q} - C_i)$ can be considered an overdamped spring that forces the shape of the elliptic vortices to a certain shape. In the limit of strong elongation the elliptic vortices can be approximated by pairs of point vortices connected via an overdamped spring with the equilibrium distance D_0 and an appropriate spring constant $\gamma/2$. This leads to the rotor model which exhibits an inverse cascade by cluster formation of same-signed vortex pairs starting from arbitrary initial conditions. See [16] for a detailed discussion.

In summary, the effect of the auxiliary field yields the extension of the point vortex model that is responsible for breaking the $\Gamma_i \rightarrow -\Gamma_i, t \rightarrow -t$ symmetry giving rise to an inverse cascade as demonstrated by the rotor model.

IX. DISCUSSION

We investigated the dynamics of the Instanton equations for the inverse cascade in two-dimensional turbulent flows. Because the dynamics of point vortices as considered in the limit of vanishing viscosity do not lead to an

inverse cascade we decided to introduce a more sophisticated model taking into account a finite spatial extension of the vortices as well as their ability to perform deformations on the basis of shear generated by distant vortices.

To this end, we derived the most probable evolution of elliptic vortices by extremalizing the Martin-Siggia-Rose action under certain assumptions. The main result is given by the evolution equations in collective coordinates (27).

We focused on the interaction of two same-signed elliptic vortices where we observed the elongation of the main axis which led to an acceleration of the relative motion. In our opinion, the combination of the elongation and the acceleration of the relative motion which leads to a reduction of the distance between two same-signed vortices is the signature of the inverse cascade. A reduced set of equations for the evolution of the vorticity field was derived which leads directly to the rotor model [16]. It exhibits an inverse cascade by cluster formation of same-signed rotors which correspond to infinitely elongated elliptic vortices.

In summary, we investigated the two-point Instanton by evaluating the interaction of two deformable same-signed vortices. We identified the underlying dynamics in accordance with [17] and [18] as elongation and thinning of elliptical shaped vortices due to the shear and hence the acceleration of the relative motion which leads to enhanced transport of energy and enstrophy. We conclude, that the Instanton calculation supports the view that elongation and thinning of small scale vortices and the resulting clustering dynamics lead to the formation of the inverse cascade in two-dimensional turbulence.

It remains an interesting task for the future to generalize our approach by modifying the ansatz. One could think of vortices with deformable main axes so that one would expect to obtain S-shaped structures or an extension to the exponential of the form $\omega \propto \exp[(\mathbf{x} C^{-1} \mathbf{x})^\alpha]$. The evaluation of the two-point statistics within the elliptical model poses an interesting challenge which consists of the evaluation of the transition probability which we sketched in the appendix (see eq. (B2)). To this end, one has to consider solutions to the Instanton equations that deviate from the point vortex vicinity and thus disobey the assumptions we made for our approach.

Appendix A: Evolution equations for elliptic vortices

We start from the Lagrangian

$$\begin{aligned} \mathcal{L} &= \sum_{i=1}^N \hat{\Gamma}_i \Gamma_i \left\{ \frac{\dot{\hat{\Gamma}}_i}{\Gamma_i} - \dot{\mathbf{x}}_i \cdot \hat{\nabla} + \frac{1}{2} \hat{\nabla} \dot{C}_i \hat{\nabla} \right. \\ &\quad \left. - i \sum_{j=1}^N \Gamma_j A_{ij}(\hat{\nabla}) - \nu \hat{\nabla} \mathcal{I} \hat{\nabla} + \frac{i}{2} \frac{\hat{\Gamma}_i}{\Gamma_i} K_i(\hat{\nabla}) \right\} W_{ii}, \end{aligned} \quad (A1)$$

which one obtains immediately from the action (26). The action and thus the Lagrangian do not depend on

the time derivative of the auxiliary field so the Euler-Lagrange equations for the variation with respect to $\hat{\omega}$ reduce to

$$\frac{\partial \mathcal{L}}{\partial \hat{f}_i} = 0 \quad (\text{A2})$$

with $\hat{f}_i \in [\hat{\Gamma}_i, \hat{\mathbf{x}}_{i_k}, \hat{C}_{i_{kl}}]$ for $k, l = (1, 2)$. The variation with respect to $\hat{\Gamma}_i$ yields

$$0 = \left\{ \frac{\dot{\Gamma}_i}{\Gamma_i} - \dot{\mathbf{x}}_i \cdot \hat{\nabla} + \frac{1}{2} \hat{\nabla} \dot{C}_i \hat{\nabla} - i \sum_{j=1}^N \Gamma_j A_{ij}(\hat{\nabla}) - \nu \hat{\nabla} \mathcal{I} \hat{\nabla} + i \frac{\hat{\Gamma}_i}{\Gamma_i} K_i(\hat{\nabla}) \right\} W_{ii}, \quad (\text{A3a})$$

the variation with respect to $\hat{\mathbf{x}}_i$ leads to

$$0 = \left\{ \frac{\dot{\Gamma}_i}{\Gamma_i} - \dot{\mathbf{x}}_i \cdot \hat{\nabla} + \frac{1}{2} \hat{\nabla} \dot{C}_i \hat{\nabla} - i \sum_{j=1}^N \Gamma_j A_{ij}(\hat{\nabla}) - \nu \hat{\nabla} \mathcal{I} \hat{\nabla} \right\} \hat{\nabla} W_{ii} \quad (\text{A3b})$$

and the variation with respect to \hat{C}_i provides

$$0 = \left\{ \frac{\dot{\Gamma}_i}{\Gamma_i} - \dot{\mathbf{x}}_i \cdot \hat{\nabla} + \frac{1}{2} \hat{\nabla} \dot{C}_i \hat{\nabla} - i \sum_{j=1}^N \Gamma_j A_{ij}(\hat{\nabla}) - \nu \hat{\nabla} \mathcal{I} \hat{\nabla} + i \frac{\hat{\Gamma}_i}{\Gamma_i} K_i(\hat{\nabla}) \right\} \hat{\nabla} \hat{\nabla}^T W_{ii}. \quad (\text{A3c})$$

We make a Gaussian ansatz for the correlation function Q according to $Q(\mathbf{k}) = q \exp[-\frac{1}{2} \mathbf{k} \tilde{Q} \mathbf{k}]$. The width of Q determines the forcing scale λ . We expand $A_{ij}(\hat{\nabla})$ and $K_i(\hat{\nabla})$ in equations (A3a)-(A3c) according to

$$A_{ij}(\hat{\nabla}) \approx i \int d^2 k' e^{-i \mathbf{k}' \cdot \mathbf{x} - \frac{1}{2} \mathbf{k}' (C_i + C_j) \mathbf{k}'} \times \left[1 + \frac{i}{2} \hat{\nabla} C_i \mathbf{k}' + \frac{i}{2} \mathbf{k}' C_i \hat{\nabla} \right] \mathbf{u}(\mathbf{k}') \cdot \hat{\nabla} \quad (\text{A4a})$$

and

$$K_i(\hat{\nabla}) \approx i q \left[1 - (\hat{\mathbf{x}}_j - \mathbf{x}_j) \cdot \hat{\nabla} + \frac{1}{2} \hat{\nabla} (\tilde{Q} + \hat{C}_i - C_i) \hat{\nabla} \right] \quad (\text{A4b})$$

and define $(\alpha_i^{(k)})$ and $(\beta_i^{(k)})$ is of k -th order in $\hat{\nabla}$

$$\begin{aligned} \alpha_i^{(1)} &= \sum_{j=1}^N \Gamma_j \mathbf{U}_{ij}(\mathbf{x}_i - \mathbf{x}_j) \cdot \hat{\nabla}, \\ \alpha_i^{(2)} &= \hat{\nabla} \sum_{j=1}^N \Gamma_j [S_{ij}(\mathbf{x}_i - \mathbf{x}_j) C_i + C_i S_{ij}^T(\mathbf{x}_i - \mathbf{x}_j)] \hat{\nabla} \\ &\quad - \hat{\nabla} \nu \mathcal{I} \hat{\nabla}, \\ \beta_i^{(0)} &= -q \frac{\hat{\Gamma}_i}{\Gamma_i}, \\ \beta_i^{(1)} &= q \frac{\hat{\Gamma}_i}{\Gamma_i} (\hat{\mathbf{x}}_i - \mathbf{x}_i) \cdot \hat{\nabla}, \\ \beta_i^{(2)} &= -q \frac{\hat{\Gamma}_i}{\Gamma_i} \frac{1}{2} \hat{\nabla} (\tilde{Q} + \hat{C}_i - C_i) \hat{\nabla}, \end{aligned}$$

where \mathbf{U}_{ij} denotes the advection

$$\mathbf{U}_{ij}(\mathbf{x}) = \int d^2 k' \mathbf{u}(\mathbf{k}') e^{-i \mathbf{k}' \cdot \mathbf{x} - \frac{1}{2} \mathbf{k}' (C_i + C_j) \mathbf{k}'} \quad (\text{A5})$$

and S_{ij} denotes the deformation

$$S_{ij}(\mathbf{x}) = -i \int d^2 k' \mathbf{u}(\mathbf{k}') \mathbf{k}'^T e^{-i \mathbf{k}' \cdot \mathbf{x} - \frac{1}{2} \mathbf{k}' (C_i + C_j) \mathbf{k}'} \quad (\text{A6})$$

which can be expressed as $\mathbf{U}_{ij}(\mathbf{x}) \hat{\nabla}^T$. We sort the obtained equations in orders of the $\hat{\nabla}$ operator and obtain with $\hat{\nabla}_{12} = \partial_{\hat{\mathbf{x}}_{i_1}} + \partial_{\hat{\mathbf{x}}_{i_2}}$

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & 0 \\ \hat{\nabla}_{12} & -1 & 0 \\ \hat{\nabla}_{12}^2 & -\hat{\nabla}_{12} & 1 \end{pmatrix} \begin{pmatrix} \frac{\hat{\Gamma}_i}{\Gamma_i} \\ \dot{\mathbf{x}}_i \cdot \hat{\nabla} \\ \frac{1}{2} \hat{\nabla} \dot{C}_i \hat{\nabla} \end{pmatrix} W_{ii} \\ &= \begin{pmatrix} -\beta_i^{(0)} \\ -\alpha_i^{(1)} - \beta_i^{(1)} \\ -\alpha_i^{(2)} - \beta_i^{(2)} - \alpha_i^{(1)} \hat{\nabla}_{12} - \beta_i^{(0)} \hat{\nabla}_{12}^2 \end{pmatrix} W_{ii}. \quad (\text{A7}) \end{aligned}$$

The matrix on the left side can be inverted and we obtain

$$\begin{pmatrix} \frac{\hat{\Gamma}_i}{\Gamma_i} \\ \dot{\mathbf{x}}_i \cdot \hat{\nabla} \\ \frac{1}{2} \hat{\nabla} \dot{C}_i \hat{\nabla} \end{pmatrix} W_{ii} = \begin{pmatrix} -\beta_i^{(0)} \\ \alpha_i^{(1)} + \beta_i^{(1)} \\ -\alpha_i^{(2)} - \beta_i^{(2)} \end{pmatrix} W_{ii}, \quad (\text{A8})$$

where we omitted terms proportional to $\hat{\nabla}_{12}$ on the right-hand side. From eq. (A8) we obtain the evolution equations

$$\begin{aligned} \dot{\Gamma}_i &= q \hat{\Gamma}_i, \\ \dot{\mathbf{x}}_i &= \sum_{j=1}^N \Gamma_j \mathbf{U}_{ij}(\mathbf{x}_i - \mathbf{x}_j) + q \frac{\hat{\Gamma}_i}{\Gamma_i} (\hat{\mathbf{x}}_i - \mathbf{x}_i(t)), \\ \dot{C}_i &= \sum_{j=1}^N \Gamma_j [S_{ij}(\mathbf{x}_i - \mathbf{x}_j) C_i + C_i S_{ij}^T(\mathbf{x}_i - \mathbf{x}_j)] \\ &\quad + q \frac{\hat{\Gamma}_i}{\Gamma_i} (\tilde{Q} + \hat{C}_i - C_i) + 2\nu \mathcal{I} \end{aligned} \quad (\text{A9})$$

for the most probable evolution of elliptic vortices.

An analogue calculation leads to the evolution equations for the elliptic structures in the auxiliary field

$$\begin{aligned}\dot{\hat{\Gamma}}_i &= 0, \\ \dot{\hat{\mathbf{x}}}_i &= \sum_{j=1}^N \Gamma_j \hat{\mathbf{U}}_{ij}(\hat{\mathbf{x}}_i - \mathbf{x}_j), \\ \dot{\hat{C}}_i &= \sum_{j=1}^N \Gamma_j \left[\hat{S}_{ij}(\hat{\mathbf{x}}_i - \mathbf{x}_j) \hat{C}_i + \hat{C}_i \hat{S}_{ij}^T(\hat{\mathbf{x}}_i - \mathbf{x}_j) \right] \\ &\quad - 2\nu \mathcal{I},\end{aligned}\tag{A10}$$

where we defined the advection and deformation for the auxiliary field

$$\begin{aligned}\hat{\mathbf{U}}_{ij}(\mathbf{x}) &= \int d^2 k' \mathbf{u}(\mathbf{k}') e^{-i\mathbf{k}' \cdot \mathbf{x} - \frac{1}{2}\mathbf{k}' \cdot (\hat{C}_i + C_j) \mathbf{k}'}, \\ \hat{S}_{ij}(\mathbf{x}) &= \hat{\mathbf{U}}_{ij}(\mathbf{x}) \overleftarrow{\nabla}^T.\end{aligned}\tag{A11}$$

To derive these equations, one has to take into account the variation of the Lagrangian with respect to Γ_i , \mathbf{x}_i and C_i as well as their time derivatives $\dot{\Gamma}_i$, $\dot{\mathbf{x}}_i$ and \dot{C}_i . Note, that we omitted higher order terms in \hat{C}_i .

Appendix B: Transition probability

With the Instanton equation for the vorticity field $\omega(\mathbf{x}, t)$ from equation (6) we obtain the extremal action

$S_e = -\frac{i}{2} \int d^2 k dt \hat{\omega}(-\mathbf{k}, t) Q(\mathbf{k}) \hat{\omega}(\mathbf{k}, t)$ where the k integration is Gaussian and leads to

$$S_e = i\pi \int_{t^*}^0 dt \left[\frac{\hat{\Gamma}_1^2}{\sqrt{\det(\hat{Q} + 2\hat{C}_1)}} + \frac{\hat{\Gamma}_2^2}{\sqrt{\det(\hat{Q} + 2\hat{C}_2)}} \right].$$

Here we omitted the overlap in accordance with our assumptions. In the Instanton approximation $\mathcal{Z} \approx e^{iS_e}$ we get an zeroth-order approximation to the transition probability density

$$f_C(\{\mathbf{x}_i^0, \Gamma_i^0\} | \{\mathbf{x}_i, \Gamma_i\}) \approx e^{iS_e} \tag{B1}$$

which depends on the shape C of the vortices at time t^* . The time t^* has to be extracted from $\Gamma_i(t) = q\hat{\Gamma}_i(t - t^*) + \Gamma_i(t^*)$. The transition probability f can be written as

$$f(\{\mathbf{x}_i^0, \Gamma_i^0\} | \{\mathbf{x}_i, \Gamma_i\}) = \int dC p(C) f_C(\{\mathbf{x}_i^0, \Gamma_i^0\} | \{\mathbf{x}_i, \Gamma_i\}) \tag{B2}$$

where $p(C)$ is a distribution of initial shapes C . The investigation of this expression is left for future work.

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